ON IMPRIMITIVE LINEAR HOMOGENEOUS GROUPS*

BY

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1. The present paper is devoted first to the proof of a theorem fundamental in the construction of imprimitive linear homogeneous groups † in a given number of variables. Then, by means of this and earlier theorems given by the author on the subject of linear groups, † JORDAN's theorem, ‡ to the effect that the order of a linear homogeneous group G in n variables is of the form λf, where f is the order of an abelian self-conjugate subgroup of G, and λ is less than a fixed number depending only upon n, is proved for imprimitive groups, a number being found that λ must divide. Finally, the principal imprimitive collineation-groups in 4 variables are found and their generating substitutions given.

THEOREM. *Either an imprimitive linear homogeneous group G can be written in monomial form,* § *or the n variables of the group can be so selected that they fall into k sets of imprimitivity of m variables each ($n = km$), permuted according to a permutation-group K in k letters, which group is transitive (in the sense of transitivity of permutation-groups). The subgroup (G') of G, corresponding to the subgroup (K') of K which leaves one letter unchanged, is primitive (in the sense used in linear homogeneous groups) in the m variables of the set corresponding to the letter that K' leaves unchanged.*

In order that G may be transitive (as a linear homogeneous group, i. e., "irreducible"), it is plainly necessary that its sets of imprimitivity contain the same number of variables, and that the permutation-group K, permuting these sets, is transitive (as a permutation-group). We shall prove that, if the

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† See articles, cited below as *Linear groups* I and II, by the author in these Transactions, vol. 4 (1903), p. 387, and vol. 5 (1904), p. 310, for definitions of terms and phrases used and for theorems employed.

‡ Journal für Mathematik, vol. 84 (1878), p. 89. JORDAN does not find a superior limit to λ. Such a limit is given for *primitive groups* by the author in *Linear groups* II. Dr. J. SCHUR has given a limit for λf for such groups in n variables, the sum of the multipliers of the substitutions of which belong to a given algebraical field (Berliner Sitzungsberichte, January, 1905).

§ This term is used by MASCHKE in American Journal of Mathematics, vol. 17 (1895), p. 168. The author used the word "semi-canonical" in *Linear groups* II, p. 313.

subgroup G' of G, corresponding to the subgroup K' of K leaving one letter fixed, is not primitive in the variables of the set corresponding to the letter that K' leaves fixed, then will G be imprimitive in a greater number of sets than k.

Let the k sets of G be $x_{11}, x_{12} \cdots x_{1m}$; $x_{21}, \cdots, x_{2m}$; $\cdots$; $x_{k1}, \cdots, x_{km}$, and let the corresponding letters of K be denoted by $y_1, y_2, \cdots, y_k$. Let K_i be the subgroup of K, leaving y_i fixed, and let G_i be the corresponding subgroup of G. The group K, being transitive, must contain a substitution (A'_2) changing y_1 into y_2, one (A'_3) changing y_1 into y_3, etc. The group G is plainly generated by G_1 and $k-1$ substitutions ($A_2, A_3, \cdots$) found among those of G corresponding to $A'_2, A'_3, \cdots$, respectively. The group obtained by erasing in G_i all the variables except $x_{i1}, x_{i2}, \cdots, x_{im}$ will be denoted by X_i.

It may readily be seen that, no matter how the group X_1 be written, the variables $x_{i1}, x_{i2}, \cdots, x_{im}$ $(i = 2, 3, \cdots, k)$ may be so selected that the substitutions $A_2, A_3, \cdots$, have the forms

$$A_2: \quad x'_{11} = x_{21},\ x'_{12} = x_{22},\ \cdots,\ x'_{1m} = x_{2m};$$

$$x'_{21} = \alpha_1 x_{11} + \alpha_2 x_{12} + \cdots + \alpha_m x_{km}, \text{ etc.};$$

$$A_3: \quad x'_{11} = x_{31},\ x'_{12} = x_{32},\ \cdots,\ x'_{1m} = x_{3m}; \text{ etc., etc.,}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Let P' be any substitution of K, and P any substitution of G taken from those corresponding to P'. If P' replaces y_i by y_j, the variables $x_{i1}, \cdots, x_{ik}$ are transformed by P in the following manner:

$$P: \begin{cases} x'_{i1} = p_{11} x_{j1} + p_{12} x_{j2} + \cdots + p_{1m} x_{jm}, \\ x'_{i2} = p_{21} x_{j1} + p_{22} x_{j2} + \cdots + p_{2m} x_{jm}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x'_{im} = p_{m1} x_{j1} + p_{m2} x_{j2} + \cdots + p_{mm} x_{jm}. \end{cases}$$

Now, if the group X_1 is not primitive, the variables $x_{11}, x_{12}, \cdots, x_{1m}$ may be supposed to have been selected so that they fall into $k_1 > 1$ sets $x_{11}, \cdots, x_{1\alpha}$; $x_{1\alpha+1}, \cdots, x_{1\beta}$; $\cdots$, all the variables of each set being by X_1 transformed into linear functions of the variables of the same set, or all into the variables of another set. Then, by building the substitution $A_i P A_j^{-1}$,* belonging to G_1:

$$x'_{11} = p_{11} x_{11} + p_{12} x_{12} + \cdots + p_{1m} x_{1m},$$

$$x'_{12} = p_{21} x_{11} + p_{22} x_{12} + \cdots + p_{2m} x_{1m},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x'_{1m} = p_{m1} x_{11} + p_{m2} x_{12} + \cdots + p_{mm} x_{1m},$$

etc.; etc.,

* For A_1 we may take the identical substitution of G.

and bearing in mind the assumption made with regard to X_1, we see that the km variables of G fall into $kk' > k$ sets

$$x_{i1}, \cdots, x_{i\alpha};\ x_{i\alpha+1}, \cdots, x_{i\beta};\ \cdots; \qquad (i=1, 2, \cdots, k),$$

which are mutually permuted by P'. The variables of G are therefore broken up into a greater number of sets of imprimitivity than k. Starting with these kk_0' sets and proceeding as above, we conclude that we must arrive ultimately at a selection of imprimitive sets for which the groups X_i are primitive or reduce to groups in one variable each, in which case G is written in monomial form. The theorem stated above is therefore proved.

It may be remarked that the writer's theorem 9, in *Linear groups* II, p. 313, follows immediately after it has been proved that any group, whose order is the power of a prime, is not primitive.

2. We shall now prove Jordan's theorem for imprimitive groups. Let us consider such a group G of order g in $n = km$ variables, the group X_i being primitive in the variables $x_{i1}, x_{i2}, \cdots, x_{im}$, if $m > k$. By § 12 of *Linear groups* II, the order of such a group X_i in m variables is of the form $\lambda_i f_i$, where f_i is the order of a self-conjugate subgroup of X_i composed of similarity substitutions, and where λ_i is a factor of a certain number that can always be calculated when m is given. Let us call this number $\phi(m)$.

The subgroup H of G corresponding to the identical substitution of K has for order h, an integral multiple of $g/k!$. This group has a subgroup F composed of substitutions which are similarity-substitutions for each of the groups $X_1, X_2, \cdots, X_k$, and whose order is an integral multiple of $h / \{\phi(m)\}^k$; i. e. an integral multiple of

$$\frac{g}{k!\,\{\phi(m)\}^k}.$$

The group F of order f is abelian, and is evidently invariant within G, and the order of the latter is of the form λf, where λ is a factor of $k!\{\phi(m)\}^k$.

3. We now pass on to the construction of the principal imprimitive collineation-groups in four variables. According to the theorem stated above, unless such a group can be written in monomial form, it must possess two sets of imprimitivity, say (x, y) and (z, u). Only the latter class of groups shall be considered here.

Let G be such a group. It is generated by an intransitive group G':

$$G': \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix},$$

where X_1 and X_2 are primitive groups in the variables (x, y) and (z, u) respectively, and a substitution R of the form

$$R:\quad x' = a_1 z + b_1 u,\quad y' = c_1 z + d_1 u,\quad z' = a_2 x + b_2 y,\quad u' = c_2 x + d_2 y.$$

Exhibiting R in the form

$$R:\ \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix} \equiv \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ c_2 & d_2 & 0 & 0 \end{bmatrix},$$

we find that $RG'R^{-1}$ takes the form

$$RG'R^{-1}:\ \begin{pmatrix} PX_2P^{-1} & 0 \\ 0 & QX_1Q^{-1} \end{pmatrix}.$$

It follows that X_1 and X_2 (as collineation-groups in two variables) are transformable one into the other by the matrices P and Q. Also, if A_1 and A_2 are corresponding substitutions of X_1 and X_2 respectively, so are PA_2P^{-1} and QA_1Q^{-1}. Moreover, we may replace R by the substitution

$$R' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} R = \begin{pmatrix} 0 & A_1P \\ A_2Q & 0 \end{pmatrix} = \begin{pmatrix} 0 & P' \\ Q' & 0 \end{pmatrix}.$$

Thus, if P is a matrix belonging to X_1, we may assume that the matrix of P' is of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Bearing these things in mind, we can construct the required groups without any theoretical difficulties, though the process will involve some labor, especially in reducing the different types obtained to certain standard forms.

We begin by determining all the groups G' possible. The groups X_1 (or X_2) are the well known tetrahedral, octahedral, and icosahedral groups, and are given in WEBER'S *Algebra*, II, 2d edition, pp. 272–287.

The types sought are generated by the following substitutions (ρ being a factor of proportionality):

	$\rho x'$	$\rho y'$	$\rho z'$	$\rho u'$	
S_1	x	$-y$	z	$-u$	
S_2	y	x	u	z	
$S_3^{(k)}$	$x+y$	$i(-x+y)$	$k(z+u)$	$ik(-z+u)$	$k^{3n}=1$.
$S_4^{(n)}$	x	y	αz	αu	α a primitive root of $\alpha^n=1$.
S_5	ix	$-iy$	z	u	
S_6	y	$-x$	z	u	
S_7	x	y	iz	$-iu$	
S_8	x	y	u	$-z$	
$S_9^{(k)}$	$x+y$	$i(-x+y)$	$(1+i)kz$	$(1+i)ku$	$k^{3n}=1$.
$S_{10}^{(l)}$	$(1+i)lx$	$(1+i)ly$	$z+u$	$i(-z+u)$	$l^{3n}=1$.
$S_{11}^{(n)}$	x	iy	βz	$i\beta u$	$\beta^n=1$.
S_{12}	x	iy	γz	γu	$\gamma^{4n}=(-1)^n$.
S_{13}	ϵx	$\epsilon^4 y$	ϵz	$\epsilon^4 u$	ϵ a primitive root of $\epsilon^5=1$.
S_{14}	$\epsilon(\omega x+y)$	$x-\omega y$	$\epsilon(\omega z+u)$	$z-\omega u$	$\omega=\epsilon+\epsilon^4$.
S_{15}	ϵx	$\epsilon^4 y$	$\epsilon^2 z$	$\epsilon^3 u$	
S_{16}	$\epsilon(\omega x+y)$	$x-\omega y$	$\epsilon^4(z-\omega u)_1$	$-\epsilon^2(\omega z+u)$	
S_{17}	ϵx	$\epsilon^4 y$	z	u	

and are as follows:

Group.	Order.	Generating Substitutions.
a	$12n$	$S_1, S_2, S_3^{(k)}, S_4^{(n)}$.
b	$4.12.2n$	$S_3^{(k)}, S_4^{(2n)}, S_5, S_6, S_7, S_8$.
c	$12.12.2n$	$S_4^{(2n)}, S_5, S_6, S_7, S_8, S_9^{(k)}, S_{10}^{(l)}$.
d	$24n$	$S_1, S_2, S_3^{(1)}, S_4^{(n)}, S_{11}^{(2n)}$.
e	$4.24.2n$	$S_3^{(1)}, S_4^{(2n)}, S_5, S_6, S_7, S_8, S_{11}^{(4n)}$.
f	$12.24.2n$	$S_4^{(2n)}, S_5, S_6, S_7, S_8, S_9^{(1)}, S_{10}^{(1)}, S_{11}^{(4n)}$.
g	$24.24.2n$	$S_4^{(2n)}, S_5, S_6, S_7, S_8, S_9^{(1)}, S_{10}^{(1)}, S_{11}^{(4n)}, S_{12}$.
h	$60n$	$S_4^{(n)}, S_{13}, S_{14}$.
h''	$60n$	$S_4^{(n)}, S_{15}, S_{16}$.
j	$60.60.2n$	$S_4^{(n)}, S_{13}, S_{14}, S_{17}$.

4. In all of these cases, the groups X_1 and X_2 (considered as collineation-groups in 2 variables) are identical. Noting what was stated at the beginning of § 3 concerning the substitution R, we see then that the matrices P and Q must in the cases a, b, c leave invariant a given tetrahedral group, and

must therefore belong to the octahedral group containing this group self-conjugately. Then, by replacing R by $R' = A \cdot R$, where A is some one of the substitutions of G', we find readily that P may be assumed to be of the form $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ or of the form $\left(\begin{smallmatrix}1 & 0\\ 0 & i\end{smallmatrix}\right)$. Under these assumptions, there will be no trouble in determining Q. In all the remaining cases we may take P in the form $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. After the resulting groups have been reduced to certain standard types, the different substitutions R will be as follows:

$$R_1: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = x, \quad \rho u' = y;$$

$$R_2: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = x, \quad \rho u' = iy;$$

$$R_3: \quad \rho x' = z, \quad \rho y' = iu, \quad \rho z' = x, \quad \rho u' = iy;$$

$$R_4: \quad \rho x' = z, \quad \rho y' = u, \quad \rho z' = y, \quad \rho u' = -x.$$

We obtain the following types of imprimitive collineation-groups in 4 variables that cannot be written in monomial form:

Group.	Order.	Generators.
1°	2.12n	R_1 and a
2°	2.12n	R_3 and a $(k=1)$
3°	2.4.12.2n	R_1 and b $(k=1)$
4°	2.4.12.2n	R_3 and b
5°	2.12.12.2n	R_1 and c $(k=l=1)$
6°	2.12.12.2n	R_3 and c
7°	2.24.n	R_1 and d
8°	2.4.24.2n	R_1 and e
9°	2.12.24.2n	R_1 and f
10°	2.12.24.2n	R_2 and f
11°	2.24.24.2n	R_1 and g
12°	2.60n	R_1 and h'
13°	2.60n	R_4 and h''
14°	2.60.60.2n	R_1 and j

The groups 1° and 12° are intransitive if $n=1$. The groups 2° and 7° are intransitive if $n=1$, $k=1$ and $\beta=1$, and 2° is of type 7° if $n=2$ and $k=1$. This is readily seen if new variables $x_1 = x+z$, $y_1 = y+u$, $z_1 = x-z$, $u_1 = y-u$ be chosen. The four groups 1°, 2°, 7°, 12° have the invariant $xu - yz = 0$ and occur therefore among those determined by Goursat in Annales scientifiques de l'École Normale supérieure, ser. 3, vol. 6 (1889), pp. 9–102. The corresponding types are there numbered XIV, XVI, XVIII, XV, XIX.

A classification of the imprimitive ("decomposable") groups which are "regular," i. e., possess the bilinear invariant $x_1 y_2 - x_2 y_1 - z_1 u_2 + z_2 u_1$, has been given by AUTONNE in Journal de Mathématiques, ser. 5, vol. 7 (1901), p. 351–394, where he gives also an extensive discussion of some of their geometrical properties.